

# 2T Physics and Quantum Mechanics

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## Abstract

We use a local scale invariance of a classical Hamiltonian and describe how to construct six different formulations of quantum mechanics in spaces with two time-like dimensions. All these six formulations have the same classical limit described by the same Hamiltonian. One of these formulations is used as a basis for a complementation of the usual quantum mechanics when in the presence of gravity.

## 1 Introduction

The wave-particle duality of matter and energy is one of the most fundamental aspects of physics. The far reaching theoretical implications of the existence of the wave-particle duality are not completely understood until now. The best known implication of this duality is that quantum mechanics can be equivalently formulated in the coordinate representation and in the momentum representation. While the coordinate representation emphasizes the particle aspect by assuming a defined position, the momentum representation is related to the wave aspect because the magnitude  $p$  of the momentum of a particle is directly related to the wave length  $\lambda$  of the associated wave by the de Broglie relation  $p = \frac{h}{\lambda}$ , where  $h$  is Planck's constant.

Some years ago it was discovered [1] that this complementarity of the descriptions in terms of coordinates and momenta of quantum mechanics can be made explicit as a classical local symmetry of an action functional describing the motion of a massless scalar relativistic particle in a space-time with an extra space-like dimension and an extra time-like dimension. For the purpose of this paper, which is to further investigate the theoretical implications of the complementarity of the wave and particle aspects of matter and energy, the interesting aspect of this new physics [2-18] with two time-like dimensions (2T physics) is that the duality of coordinates and momenta appears already at the classical level, and this makes it easier to follow its implications because the quantum mechanical ordering ambiguities are absent. In this paper we use the local indistinguishability of coordinate and momentum in 2T physics to suggest a complementation of the basic equations of quantum mechanics.

In a previous paper [19], we presented a finite local scale invariance of the 2T physics Hamiltonian and showed how this local invariance can be used to relate the  $d + 2$  dimensional Minkowski space of 2T physics to a Riemannian space of the same dimensionality. Although changing from a flat space to a curved position dependent space using a local invariance is already an interesting observation, it is not the only one. The finite local scale invariance of the 2T Hamiltonian also associates to the  $d + 2$  dimensional Minkowski space of 2T physics another  $d + 2$  dimensional Riemannian space where the geometry is described by a momentum dependent tensor. More surprising is that the Hamiltonian equations of motion are identical in these three spaces.

In the usual one-time (1T) physics, position dependent metric tensors play an important role in the most general position space formulation of quantum mechanics [20]. In this general formulation, these tensors appear in the spectral decomposition of the unity, define the correct integration measure for the inner product and are present in the most general expression of the position matrix elements for self adjoint momentum operators in position space [20]

$$\begin{aligned} \langle x | \hat{p}_\alpha | x \rangle &= \frac{i\hbar}{g^{\frac{1}{4}}(x)} \frac{\partial}{\partial x^\alpha} \left[ \frac{1}{g^{\frac{1}{4}}(x)} \delta^n(x - x') \right] \\ &+ \frac{1}{\sqrt{g(x)}} A_\alpha(x) \delta^n(x - x') \end{aligned} \quad (1.1)$$

where  $g(x) = \det g_{\alpha\beta}(x)$  and  $\alpha, \beta = 1, \dots, n$ . Since quantum mechanics can be equivalently formulated in the position or in the momentum representation, the appearance in 2T physics of a momentum dependent metric tensor may be considered as an indication that the momentum space versions of quantum mechanical equations such as (1.1) and others are still lacking. The construction of these momentum space equations is one of the motivations for this paper.

As can be seen in (1.1), the other central object in the general position space formulation of quantum mechanics described in [20] is the vector field  $A_\alpha(x)$ . It has a vanishing strength tensor,

$$F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta} = 0 \quad (1.2)$$

and because of this condition it defines a section of a flat U(1) bundle over the position space. The vector field is present only if the position space has a non-trivial topology. In position spaces with trivial topology  $A_\alpha(x)$  can always be gauged away [20].

In this paper we use the results of [19], together with new results that are presented here, to show how one can extend to momentum space the general position space formulation of quantum mechanics described in [20]. As we will see here, in addition to the concept of a momentum dependent metric tensor which was found necessary in [19], this paper evidences the need for the concept of a momentum dependent vector field. Combining these two concepts we can write down the basic equations for a formulation of quantum mechanics

in total agreement with the wave-particle duality. This paper also gives relevant contributions to the development of 2T physics with vector fields. In addition to the local scale invariance of the classical Hamiltonian equations of motion in the case when  $A_M = A_M(X)$  that was presented in [19], this paper also points out that the classical Hamiltonian equations of motion are invariant under local scale transformations in the case when  $A_M = A_M(P)$ . For each of the symmetries of the 2T action in the case when  $A_M = A_M(X)$  that were presented in [19], this paper also presents the corresponding symmetry with  $A_M = A_M(P)$ . This existence of the same symmetries of the 2T action in position space and in momentum space is what gives support for the new quantum mechanical equations we write here.

The paper is organized as follows. In section two we briefly review how we can use a local scale invariance of the massless scalar relativistic particle Hamiltonian to introduce a new bracket structure in phase space. These new brackets are the classical analogues of the Snyder commutators [21] in the case where the noncommutativity parameter is  $\theta = 1$ . The Snyder commutators were derived in 1947 in a projective geometry approach to the de Sitter (dS) space in the momentum representation and are considered here as the first evidence for momentum dependent metric tensors in quantum mechanics in the presence of gravity.

The case of momentum dependent metric tensors was not included in the general formulation of quantum mechanics presented in [20]. However, the necessity for this kind of tensor field in quantum mechanics is clearly suggested in the most general expression for the wave function  $\langle x | p \rangle$  obtained in [20]. In this paper we adopt the point of view that momentum dependent tensor fields are necessary in quantum mechanics in the presence of gravity as a consequence of the wave-particle duality and in section three we briefly review how we can use the local indistinguishability of position and momentum in 2T physics to show that momentum dependent metric tensors also have a natural existence in  $d + 2$  dimensions.

Section four presents the basic equations of a formulation of quantum mechanics that completely incorporates the wave-particle duality. This is done by introducing the corresponding momentum space expressions of the position space expressions obtained in [20]. A difficulty that appears in this formulation of quantum mechanics is that we also need the concept of a momentum dependent vector field. As a basis for this concept, in section five we present the action that describes in a unified way 2T physics with position dependent or momentum dependent vector fields. For each symmetry of our action in position space we present the corresponding symmetry in momentum space. We also discuss the scale invariance of the equations of motion when  $A_M = A_M(X)$  and when  $A_M = A_M(P)$  and conclude that there are six possible equivalent formulations of quantum mechanics that have the same classical Hamiltonian limit described by 2T physics. Further concluding remarks appear in section six.

## 2 Massless Relativistic Particles

In this section we briefly review how the classical analogues of the Snyder commutators, obtained in 1947 for the dS space in the momentum representation, can be derived in massless scalar relativistic particle theory using a local scale invariance of the Hamiltonian. For details the reader should see [19].

A massless scalar relativistic particle in a  $d$ -dimensional Minkowski space-time with signature  $(d-1, 1)$  is described by the action

$$S = \frac{1}{2} \int_{\tau_i}^{\tau_f} d\tau \lambda^{-1} \dot{x}^2 \quad (2.1)$$

where  $\lambda(\tau)$  is an auxiliary variable,  $x^\mu = x^\mu(\tau)$ ,  $\dot{x}^2 = \dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu}$  and  $\eta_{\mu\nu}$  is the flat Minkowski metric. A dot denotes derivatives with respect to the parameter  $\tau$ . Action (2.1) is invariant under the local infinitesimal reparameterizations

$$\delta x_\mu = \alpha(\tau) \dot{x}_\mu \quad \delta \lambda = \frac{d}{d\tau} [\alpha(\tau) \lambda]$$

and therefore describes gravity on the world-line. In the transition to the Hamiltonian formalism action (2.1) gives the canonical momenta

$$p_\lambda = 0 \quad (2.2)$$

$$p_\mu = \frac{\dot{x}_\mu}{\lambda} \quad (2.3)$$

and the canonical Hamiltonian

$$H = \frac{1}{2} \lambda p^2 \quad (2.4)$$

Equation (2.2) is a primary constraint [22]. Introducing the Lagrange multiplier  $\xi(\tau)$  for this constraint we can write the Dirac Hamiltonian

$$H_D = \frac{1}{2} \lambda p^2 + \xi p_\lambda \quad (2.5)$$

Requiring the dynamical stability of constraint (2.2),  $\dot{p}_\lambda = \{p_\lambda, H_D\} = 0$ , and using the Poisson bracket  $\{\lambda, p_\lambda\} = 1$ , we obtain the secondary constraint

$$\phi = \frac{1}{2} p^2 \approx 0 \quad (2.6)$$

Constraints (2.2) and (2.6) have vanishing Poisson bracket and are therefore first class constraints [22]. Constraint (2.2) generates translations in the arbitrary variable  $\lambda(\tau)$  and can be dropped from the formalism. The notation  $\approx$  means that  $\phi$  *weakly vanishes* [23]. Weak equalities hold over the entire phase space and can be turned into strong equalities on the constraint surface.

Action (2.1) can be rewritten in Hamiltonian form as

$$S = \int_{\tau_i}^{\tau_f} d\tau (\dot{x} \cdot p - \frac{1}{2} \lambda p^2) \quad (2.7)$$

The first class constraint (2.6) generates the gauge transformations

$$\delta x_\mu = \epsilon(\tau)\{x_\mu, \phi\} = \epsilon(\tau)p_\mu \quad (2.8a)$$

$$\delta p_\mu = \epsilon(\tau)\{p_\mu, \phi\} = 0 \quad (2.8b)$$

$$\delta\lambda = \dot{\epsilon}(\tau) \quad (2.8c)$$

computed in terms of the Poisson brackets

$$\{x_\mu, x_\nu\} = 0 \quad \{p_\mu, p_\nu\} = 0 \quad \{x_\mu, p_\nu\} = \eta_{\mu\nu} \quad (2.9)$$

under which action (2.7) transforms as

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \frac{d}{d\tau}(\epsilon\phi) \quad (2.10)$$

Since the interval  $(\tau_i, \tau_f)$  is arbitrary, action (2.7) is invariant under transformation (2.8) and the quantity  $Q = \epsilon\phi$  can be interpreted as the conserved Hamiltonian Noether charge or as the generator of the gauge transformations (2.8), depending on whether the equations of motion are satisfied or not [24]. This property of Hamiltonian Noether charges  $Q$  will be used to confirm some of the results contained in this paper.

The gravitational field, regarded as a gauge field [25], can correspond to several symmetry groups: 1) the general covariant group, 2) the local Lorentz group, and 3) the group of scale transformations of the interval. In the first case the properties of the gravitational field are determined by the properties of the metric tensor, and this gives the usual Einstein theory. In the second case they are determined by the properties of the Ricci connection coefficients and this leads to equations of the fourth order. In the third case, it is assumed that the source of the gravitational field is the trace of the energy-momentum tensor and that the carriers are scalar particles [25]. In agreement with the third point of view, the massless particle Hamiltonian (2.4) is invariant under the finite local scale transformations

$$p_\mu \rightarrow \exp\{-\beta(\tau)\}p_\mu \quad (2.11a)$$

$$\lambda \rightarrow \exp\{2\beta(\tau)\}\lambda \quad (2.11b)$$

where  $\beta(\tau)$  is an arbitrary scalar function. From equation (2.3) for the canonical momentum we find that  $x^\mu$  transforms as

$$x_\mu \rightarrow \exp\{\beta(\tau)\}x_\mu \quad (2.11c)$$

when  $p_\mu$  transforms as in (2.11a). We can use the arbitrary character of the function  $\beta$  in the local scale invariance (2.11), together with the first class property of constraint (2.6), to change to a bracket structure different from the usual Poisson brackets (2.9). The simplest possibility is to choose  $\beta = \frac{1}{2}p^2$ . In this gauge the phase space has the bracket structure [19]

$$\{p_\mu, p_\nu\} \approx 0 \quad (2.12a)$$

$$\{x_\mu, p_\nu\} \approx \eta_{\mu\nu} - p_\mu p_\nu \quad (2.12b)$$

$$\{x_\mu, x_\nu\} \approx -M_{\mu\nu} \quad (2.12c)$$

where  $M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$  is the generator of Lorentz transformations. It can be verified that all Jacobi identities among the canonical variables still close if we use the brackets (2.12) instead of the Poisson brackets (2.9). In the transition to the quantized theory using the correspondence principle rule that [commutator]= $i\hbar$ {bracket}, the brackets (2.12) will reproduce the Snyder commutators [21] for the dS space in the momentum representation and with noncommutativity parameter  $\theta = 1$ .

In the presence of gravity and at length scales near the Planck length, the fundamental commutator  $[x_\mu, p_\nu] = i\hbar\eta_{\mu\nu}$  of quantum mechanics must be replaced [26] by a more general commutator  $[x_\mu, p_\nu] = i\hbar g_{\mu\nu}$ . This is because the large amounts of relativistic momentum involved in the quantum measurement processes necessarily modify the space-time geometry at these length scales [26].  $g_{\mu\nu}$  is in principle a function of the positions, but this paper calls attention to the fact that  $g_{\mu\nu}$  can also be a momentum dependent function. This can be seen in the bracket (2.12b) we derived for the massless relativistic particle. This can also be seen in the old Snyder commutators for the dS space in the momentum representation.

Hamiltonian (2.4) generates the classical equations of motion

$$\dot{x}_\mu = \{x_\mu, H\} = \lambda p_\mu \quad (2.13a)$$

$$\dot{p}_\mu = \{p_\mu, H\} = 0 \quad (2.13b)$$

computed in terms of the Poisson brackets (2.9). If we now change to the momentum dependent background

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} - p_\mu p_\nu \quad (2.14)$$

implied by bracket (2.12b), the new Hamiltonian  $\bar{H}$  in this background is given by

$$\bar{H} = H - 2\lambda\phi^2 \quad (2.15)$$

The Hamiltonian (2.15) in the background (2.14) differs from (2.4) by a term that is quadratic in constraint (2.6). This term can be dropped and the new Hamiltonian in the background (2.14) is identical to (2.4) in Minkowski space.

Although the Hamiltonians are identical, in the background (2.14) the Poisson brackets (2.9) are no longer valid. They must be replaced by brackets (2.12). Computing the equations of motion using the Hamiltonian (2.4) and brackets (2.12) we find

$$\dot{x}_\mu = \{x_\mu, H\} = \lambda p_\mu - 2\lambda p_\mu \phi \quad (2.16a)$$

$$\dot{p}_\mu = \{p_\mu, H\} = 0 \quad (2.16b)$$

We see that the equations of motion in the background (2.14) differ from (2.13) by a term that is linear in the constraint  $\phi$ . Again, this term can be dropped. From these observations we may conclude that, at the classical level, the massless particle Hamiltonian dynamics in the momentum space background (2.14) is indistinguishable from the Hamiltonian dynamics in Minkowski space.

### 3 2T Physics

In this section we consider how the position space and momentum space higher-dimensional extensions of brackets (2.12) can be obtained from a local scale invariance of the 2T Hamiltonian. We also consider the invariance of the classical equations of motion in the corresponding backgrounds.

The construction of 2T physics [1-18] is based on the introduction of a new gauge invariance in phase space by gauging the duality of the quantum commutator  $[X_M, P_N] = i\hbar\eta_{MN}$ . This procedure leads to a symplectic  $Sp(2, R)$  gauge theory. To remove the distinction between position and momentum we rename them  $X_1^M = X^M(\tau)$  and  $X_2^M = P^M(\tau)$  and define the doublet  $X_i^M(\tau) = (X_1^M, X_2^M)$ . The local  $Sp(2, R)$  symmetry acts as

$$\delta X_i^M(\tau) = \epsilon_{ik}\omega^{kl}(\tau)X_l^M(\tau) \quad (3.1)$$

$\omega^{ij}(\tau)$  is a symmetric matrix containing three local parameters and  $\epsilon_{ij}$  is the Levi-Civita symbol that serves to raise or lower indices. The  $Sp(2, R)$  gauge field  $A^{ij}$  is symmetric in  $(i, j)$  and transforms as

$$\delta A^{ij} = \partial_\tau \omega^{ij} + \omega^{ik}\epsilon_{kl}A^{lj} + \omega^{jk}\epsilon_{kl}A^{il} \quad (3.2)$$

The covariant derivative is

$$D_\tau X_i^M = \partial_\tau X_i^M - \epsilon_{ik}A^{kl}X_l^M \quad (3.3)$$

An action invariant under the  $Sp(2, R)$  gauge symmetry is

$$S = \frac{1}{2} \int d\tau (D_\tau X_i^M) \epsilon^{ij} X_j^N \eta_{MN} \quad (3.4a)$$

After an integration by parts this action can be written as

$$\begin{aligned} S &= \int d\tau (\partial_\tau X_1^M X_2^N - \frac{1}{2} A^{ij} X_i^M X_j^N) \eta_{MN} \\ &= \int d\tau [\dot{X} \cdot P - (\frac{1}{2} \lambda_1 P^2 + \lambda_2 X \cdot P + \frac{1}{2} \lambda_3 X^2)] \end{aligned} \quad (3.4b)$$

where  $A^{11} = \lambda_3$ ,  $A^{12} = A^{21} = \lambda_2$ ,  $A^{22} = \lambda_1$  and the canonical Hamiltonian is

$$H = \frac{1}{2} \lambda_1 P^2 + \lambda_2 X \cdot P + \frac{1}{2} \lambda_3 X^2 \quad (3.5)$$

The equations of motion for the  $\lambda$ 's give the first class constraints

$$\phi_1 = \frac{1}{2} P^2 \approx 0 \quad (3.6)$$

$$\phi_2 = X \cdot P \approx 0 \quad (3.7)$$

$$\phi_3 = \frac{1}{2} X^2 \approx 0 \quad (3.8)$$

Constraints (3.6)-(3.8), as well as evidences of 2T physics, were independently obtained in [27]. Equations (3.6) and (3.8) can be interpreted as constraints only if the hypersurfaces  $P_0 = P_1 = \dots = P_{d+1} = 0$  and  $X_0 = X_1 = \dots = X_{d+1} = 0$  are removed from phase space [27]. Only in this case we have a consistent Hamiltonian formalism defined over a regular [23] constraint surface. This removal of the origin of phase space induces a non-trivial phase space topology. In the case of the usual position space of 1T physics, a non-trivial topology is associated with the presence of a position dependent vector field in the quantized theory [20]. This vector field has a vanishing strength tensor and defines a section of a flat  $U(1)$  bundle over the position space [20]. The presence of general vector fields in 2T physics will be considered in section five. The case of position dependent vector fields with non-vanishing strength tensors was first discussed in [5].

If we consider the usual Minkowski metric as the background space, we find that the surface defined by the constraint equations (3.6)-(3.8) is trivial. The metrics giving a non-trivial constraint surface, preserving the unitarity of the theory, and avoiding the ghost problem are the metrics with two time-like dimensions [1-18]. For the purposes of this paper it is best start working in a Minkowski space with signature  $(d, 2)$ . Action (3.4b) is the  $(d+2)$ -dimensional extension of the  $d$ -dimensional massless particle action (2.7). Action (3.4b) describes conformal gravity on the world-line [28,29,1].

We now introduce the Poisson brackets

$$\{P_M, P_N\} = 0 \quad \{X_M, X_N\} = 0 \quad \{X_M, P_N\} = \eta_{MN} \quad (3.9)$$

It can then be checked that action (3.4b) is invariant under Lorentz  $SO(d, 2)$  transformations with generator  $L_{MN} = X_M P_N - X_N P_M$

$$\delta X_M = \frac{1}{2} \omega_{RS} \{L_{RS}, X_M\} = \omega_{MR} X_R \quad (3.10a)$$

$$\delta P_M = \frac{1}{2} \omega_{RS} \{L_{RS}, P_M\} = \omega_{MR} P_R \quad (3.10b)$$

$$\delta \lambda_\alpha = 0, \quad \alpha = 1, 2, 3 \quad (3.10c)$$

under which  $\delta S = 0$ . The  $L_{MN}$  are gauge invariant because they have vanishing brackets with constraints (3.6)-(3.8).

The first class constraints (3.6)-(3.8) generate the local transformations

$$\delta X_M = \epsilon_\alpha(\tau) \{X_M, \phi_\alpha\} = \epsilon_1 P_M + \epsilon_2 X_M \quad (3.11a)$$

$$\delta P_M = \epsilon_\alpha(\tau) \{P_M, \phi_\alpha\} = -\epsilon_2 P_M - \epsilon_3 X_M \quad (3.11b)$$

$$\delta \lambda_1 = \dot{\epsilon}_1 + 2\epsilon_2 \lambda_1 - 2\epsilon_1 \lambda_2 \quad (3.11c)$$

$$\delta \lambda_2 = \dot{\epsilon}_2 + \epsilon_3 \lambda_1 - \epsilon_1 \lambda_3 \quad (3.11d)$$

$$\delta \lambda_3 = \dot{\epsilon}_3 + 2\epsilon_3 \lambda_2 - 2\epsilon_2 \lambda_3 \quad (3.11e)$$



under which

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \frac{d}{d\tau} (\epsilon_\alpha \phi_\alpha) \quad (3.12)$$

Similarly to the massless particle case, since the interval  $(\tau_i, \tau_f)$  is arbitrary, the quantity  $Q = \epsilon_\alpha \phi_\alpha$ , with  $\alpha = 1, 2, 3$ , can be interpreted as the conserved Hamiltonian Noether charge or as the generator of the local transformations (3.11), depending on whether the equations of motion are satisfied or not [24].

The 2T Hamiltonian (3.5) is invariant under the finite local scale transformations

$$X_M \rightarrow \exp\{\beta(\tau)\} X_M \quad (3.13a)$$

$$P_M \rightarrow \exp\{-\beta(\tau)\} P_M \quad (3.13b)$$

$$\lambda_1 \rightarrow \exp\{2\beta(\tau)\} \lambda_1 \quad (3.13c)$$

$$\lambda_2 \rightarrow \lambda_2 \quad (3.13d)$$

$$\lambda_3 \rightarrow \exp\{-2\beta(\tau)\} \lambda_3 \quad (3.13e)$$

where  $\beta(\tau)$  is an arbitrary scalar function. Now we can use the scale transformation (3.13), together with the first class property of constraints (3.6)-(3.8), and by choosing the arbitrary function to be  $\beta = \frac{1}{2}P^2$ , arrive at the brackets [19]

$$\{P_M, P_N\} \approx 0 \quad (3.14a)$$

$$\{X_M, P_N\} \approx \eta_{MN} - P_M P_N \quad (3.14b)$$

$$\{X_M, X_N\} \approx -L_{MN} \quad (3.14c)$$

Brackets (3.14) are the  $(d+2)$ -dimensional extensions of the  $d$ -dimensional momentum space brackets (2.12) we found for the massless particle in the previous section. By choosing the arbitrary scalar function  $\beta$  to be a function of  $P_M(\tau)$ , we arrived at the momentum space brackets (3.14). But in 2T physics momentum and position are indistinguishable variables. So, in 2T physics, there must exist a position space version of brackets (3.14). This position space version can be reached by choosing  $\beta = \frac{1}{2}X^2$  in transformation (3.13). Using again the first class property of constraints (3.6)-(3.8), we arrive at the brackets [19]

$$\{P_M, P_N\} \approx L_{MN} \quad (3.15a)$$

$$\{X_M, P_N\} \approx \eta_{MN} + X_M X_N \quad (3.15b)$$

$$\{X_M, X_N\} \approx 0 \quad (3.15c)$$

Notice that brackets (3.15) can not be obtained from brackets (3.14) by performing the duality transformation  $X_M \rightarrow P_M$ ,  $P_M \rightarrow -X_M$  which leaves the quantum commutator  $[X_M, P_N] = i\hbar\eta_{MN}$  invariant. This duality does not allow us to perform a transition from brackets (3.14) to brackets (3.15). The transition from (3.14) to (3.15) involves the local scale invariance (3.13) of the 2T Hamiltonian (3.5). Therefore this transition involves conformal gravity on the world-line.

If we choose  $\beta(\tau) = 0$  in transformation (3.13) we obtain the Poisson brackets (3.9). Working with the Poisson brackets (3.9), or with brackets (3.14), or with brackets (3.15) is a matter of gauge choice and therefore these three sets of brackets must lead to equivalent results at the classical level. This can be easily verified to be true. Using the Poisson brackets (3.9) we find that the 2T Hamiltonian (3.5) generates the classical equations of motion

$$\dot{X}_M = \{X_M, H\} = \lambda_1 P_M + \lambda_2 X_M \quad (3.16a)$$

$$\dot{P}_M = \{P_M, H\} = -\lambda_2 P_M - \lambda_3 X_M \quad (3.16b)$$

After dropping terms proportional to the first class constraints (3.6)-(3.8), we find that the 2T Hamiltonian (3.5) and the equations of motion (3.16) remain invariant if we change to the background

$$\bar{G}_{MN} = \eta_{MN} - P_M P_N \quad (3.17)$$

and simultaneously replace the Poisson brackets (3.9) by brackets (3.14). The 2T Hamiltonian (3.5) and the equations of motion (3.16) also remain invariant if we change to the background

$$G_{MN} = \eta_{MN} + X_M X_N \quad (3.18)$$

and simultaneously replace the Poisson brackets (3.9) by brackets (3.15). As a consequence of the local scale invariance (3.13) there are three equivalent classical Hamiltonian formulations of 2T physics. The first formulation is in the usual Minkowski space of 2T physics using the standard Poisson brackets (3.9). The second formulation is in the momentum space background (3.17) using brackets (3.14). The third formulation is in the position space background (3.18) using brackets (3.15).

## 4 Extended Quantum Mechanics

The results of the previous sections bring with them the possibility of a deeper insight into the formal structure of quantum mechanics. The idea is to explicitly incorporate into quantum mechanics the indistinguishability of position and momentum of 2T physics. This can be done without the need of extra dimensions. To extend quantum mechanics we must introduce an additional assumption between assumptions A1 and A2 of reference [20]. The formulation of quantum mechanics we present in this section is based in three assumptions, of which the first and the third ones are identical to A1 and A2 in [20]. Our assumptions are

1) There exists a basis  $|x\rangle$  of the position space which is spanned by the eigenvalues of the position operators  $\hat{x}^\alpha$  ( $\alpha = 1, 2, \dots, n$ ), whose domain of eigenvalues coincides with all the possible values of the coordinates  $x^\alpha$  parameterizing the position space  $M(x)$ ,

$$\hat{x}^\alpha |x\rangle = x^\alpha |x\rangle \quad , \quad \{x^\alpha\} \in M(x)$$

2) There exists a basis  $|p\rangle$  of the momentum space which is spanned by the eigenvalues of the momentum operators  $\hat{p}_\alpha$  ( $\alpha = 1, 2, \dots, n$ ), whose domain of eigenvalues coincides with all the possible values of the momenta  $p_\alpha$  parameterizing the momentum space  $D(p)$ ,

$$\hat{p}_\alpha |p\rangle = p_\alpha |p\rangle \quad , \quad \{p_\alpha\} \in D(p)$$

3) The representation spaces of the algebra are endowed with a Hermitian positive definite inner product  $\langle \cdot | \cdot \rangle$  for which the operators  $\hat{x}^\alpha$  and  $\hat{p}_\alpha$  are self-adjoint.

Now consider quantum mechanics in position space. The construction of quantum mechanics describing the diffeomorphic-covariant representations of the Heisenberg algebra in terms of topological classes of a flat U(1) bundle over position space has the parameterization [20] of the inner product  $\langle x | x' \rangle$

$$\langle x | x' \rangle = \frac{1}{\sqrt{g(x)}} \delta^n(x - x') \quad (4.1)$$

where  $g(x)$  is an arbitrary positive definite function defined over the position space  $M$ . For a Riemannian manifold the natural choice [20] for  $g$  is the determinant of the metric tensor,  $g = \det g_{\alpha\beta}(x)$ . Position dependent metric tensors naturally appear in this formulation of quantum mechanics.

Equation (4.1) implies the spectral decomposition [20] of the identity operator in the position eigenbasis  $|x\rangle$

$$1 = \int_M d^n x \sqrt{g(x)} |x\rangle \langle x| \quad (4.2)$$

which in turn leads to the position space wave function representations  $\psi(x) = \langle x | \psi \rangle$  and  $\langle \psi | x \rangle = \langle x | \psi \rangle^* = \psi^*(x)$  of any state  $|\psi\rangle$  belonging to the Heisenberg algebra representation space,

$$|\psi\rangle = \int_M d^n x \sqrt{g(x)} \psi(x) |x\rangle \quad (4.3)$$

$$\langle \psi | = \int_M d^n x \sqrt{g(x)} \psi^*(x) \langle x | \quad (4.4)$$

The inner product of two states  $|\psi\rangle$  and  $|\varphi\rangle$  is then given in terms of their position space wave functions  $\psi(x)$  and  $\varphi(x)$  as

$$\langle \psi | \varphi \rangle = \int_M d^n x \sqrt{g(x)} \psi^*(x) \varphi(x) \quad (4.5)$$

The most general position space wave function representations of the position and momentum operators are [20]

$$\langle x | \hat{x}_\alpha | \psi \rangle = x_\alpha \langle x | \psi \rangle = x_\alpha \psi(x) \quad (4.6a)$$

$$\langle x | \hat{p}_\alpha | \psi \rangle = \frac{-i\hbar}{g^{1/4}(x)} \left[ \frac{\partial}{\partial x^\alpha} + \frac{i}{\hbar} A_\alpha(x) \right] g^{1/4}(x) \psi(x) \quad (4.6b)$$

The vector field  $A_\alpha(x)$  is present only in the case of topologically non-trivial position spaces [20]. It has a vanishing strength tensor  $F_{\alpha\beta}$  as given by (1.2) and is related to arbitrary local phase transformations of the position eigenvectors

$$| x' \rangle = e^{\frac{i}{\hbar} \chi(x)} | x \rangle \quad (4.7a)$$

when

$$A'_\alpha(x) = A_\alpha(x) + \frac{\partial \chi(x)}{\partial x^\alpha} \quad (4.7b)$$

where  $\chi(x)$  is an arbitrary scalar function. From the above equations we see that in Riemannian spaces, where  $g(x) = \det g_{\alpha\beta}$ , position dependent metric tensors play a central role in this formulation of quantum mechanics. The other central object in this formulation is the vector field  $A_\alpha(x)$  of vanishing strength tensor.

Now we consider quantum mechanics in momentum space and extend the construction in [20]. The normalization of the momentum eigenstates is parameterized according to [20]

$$\langle p | p' \rangle = \frac{1}{\sqrt{h(p)}} \delta^n(p - p') \quad (4.8)$$

where  $h(p)$  is an arbitrary positive definite function defined over the domain  $D(p)$  of the momentum eigenvalues. The authors in [20] do not go beyond this point and do not consider the possible forms of the function  $h(p)$ . However, from our experience with the massless scalar relativistic particle in section two and with 2T physics in section three, we may expect that in a momentum space with a non-trivial geometry, such as the de Sitter space in Snyder's momentum space approach, a natural choice is  $h(p) = \det \bar{g}_{\mu\nu}$ , where  $\bar{g}_{\mu\nu}$  is given by equation (2.14). This leads to the idea that the wave-particle duality may require that momentum dependent metric tensors be present in the most general momentum space formulation of quantum mechanics in the presence of gravity, in the same way as position dependent metric tensors are present in the most general position space formulation of quantum mechanics in the presence of gravity described in [20]. But we also need another central object to complete the formal structure of quantum mechanics. We need the concept of a momentum dependent vector field  $A_\alpha(p)$  with a vanishing strength tensor in momentum space.

As a consequence of (4.8) and of our second assumption, we have the spectral decomposition of the identity operator in the momentum eigenbasis  $| p \rangle$

$$1 = \int_{D(p)} d^n p \sqrt{h(p)} | p \rangle \langle p | \quad (4.9)$$

This leads to the momentum space wave functions  $\psi(p) = \langle p | \psi \rangle$  and  $\langle \psi | p \rangle = \langle p | \psi \rangle^* = \psi^*(p)$  of any state  $| \psi \rangle$  belonging to the Heisenberg algebra

representation space

$$|\psi\rangle = \int_{D(p)} d^n p \sqrt{h(p)} \psi(p) |p\rangle \quad (4.10a)$$

$$\langle\psi| = \int_{D(p)} d^n p \sqrt{h(p)} \psi^*(p) \langle p| \quad (4.10b)$$

The inner product of two states  $|\psi\rangle$  and  $|\varphi\rangle$  is given in terms of their momentum space wave functions  $\psi(p)$  and  $\varphi(p)$  as

$$\langle\psi|\varphi\rangle = \int_{D(p)} d^n p \sqrt{h(p)} \psi^*(p) \varphi(p) \quad (4.11)$$

The most general wave function  $\langle x|p\rangle$  is given by [20]

$$\langle x|p\rangle = \frac{e^{i\varphi(x_0,p)}}{(2\pi\hbar)^{\frac{n}{2}}} \frac{\Omega[P(x_0 \rightarrow x)]}{g^{\frac{1}{4}}(x)h^{\frac{1}{4}}(p)} e^{\frac{i}{\hbar}(x-x_0)\cdot p} \quad (4.12)$$

$\varphi(x_0, p)$  is a specific but otherwise arbitrary real function and  $\Omega[P(x_0 \rightarrow x)]$  is the path ordered U(1) holonomy along the path  $P(x_0 \rightarrow x)$ . Notice that  $g(x)$  and  $h(p)$  are both necessary because they simultaneously appear in the most general wave function (4.12). The wave function (4.12) generalizes in a transparent manner the usual plane wave solutions of application to the trivial representation of the Heisenberg algebra with  $A_\alpha(x) = 0$  and with the choices  $g(x) = 1$  and  $h(p) = 1$ .

Now we point out that the wave-particle duality can be made explicit in quantum mechanics if we introduce the equations

$$\langle p|\hat{p}_\alpha|\psi\rangle = p_\alpha \langle p|\psi\rangle = p_\alpha \psi(p) \quad (4.13a)$$

$$\langle p|\hat{x}_\alpha|\psi\rangle = \frac{i\hbar}{h^{1/4}(p)} \left[ \frac{\partial}{\partial p^\alpha} + \frac{i}{\hbar} A_\alpha(p) \right] h^{1/4}(p) \psi(p) \quad (4.13b)$$

which are the momentum space correspondents of equations (4.6). The vector field  $A_\alpha(p)$  has a vanishing strength tensor in momentum space,

$$\bar{F}_{\alpha\beta} = \frac{\partial A_\beta}{\partial p^\alpha} - \frac{\partial A_\alpha}{\partial p^\beta} = 0 \quad (4.14)$$

and is related to arbitrary local phase transformations of the momentum eigenvectors

$$|p'\rangle = e^{\frac{i}{\hbar}\gamma(p)} |p\rangle \quad (4.15a)$$

when

$$A'_\alpha(p) = A_\alpha(p) + \frac{\partial \gamma(p)}{\partial p^\alpha} \quad (4.15b)$$

where  $\gamma(p)$  is an arbitrary scalar function. Equations (4.15) are the momentum space correspondents of equations (4.7). Now we need one evidence that the

complementation of the basic equations of quantum mechanics we proposed in this section is really necessary. The best place to search for this evidence is in 2T physics. First, because 2T physics is based on the local indistinguishability of position and momentum. And second, because it is now clear that 1T physics is embedded in 2T physics [18].

## 5 2T Physics and Quantum Mechanics

In section three we saw that position dependent and momentum dependent metric tensors naturally appear in 2T physics. To show that position dependent and momentum dependent vector fields also have a natural existence and a unified description in 2T physics, we first modify the 2T Hamiltonian (3.5) according to the usual minimal coupling prescription to position dependent vector fields,  $P_M \rightarrow P_M - A_M(X)$ . Action (3.4b) then becomes

$$S = \int d\tau \{ \dot{X} \cdot P - [\frac{1}{2}\lambda_1(P - A)^2 + \lambda_2 X \cdot (P - A) + \frac{1}{2}\lambda_3 X^2] \} \quad (5.1)$$

The equations of motion for the Lagrange multipliers now give the constraints

$$\phi_1 = \frac{1}{2}(P - A)^2 \approx 0 \quad (5.2)$$

$$\phi_2 = X \cdot (P - A) \approx 0 \quad (5.3)$$

$$\phi_3 = \frac{1}{2}X^2 \approx 0 \quad (5.4)$$

The Poisson brackets between the canonical variables and the vector field  $A_M(X)$  are

$$\{X_M, A_N\} = 0 \quad (5.5a)$$

$$\{P_M, A_N\} = -\frac{\partial A_N}{\partial X^M} \quad (5.5b)$$

$$\{A_M, A_N\} = 0 \quad (5.5c)$$

Computing the algebra of constraints (5.2)-(5.4) using the Poisson brackets (3.9) and (5.5) we find the expressions

$$\{\phi_1, \phi_1\} = (P^M - A^M)F_{MN}(P^N - A^N) \quad (5.6a)$$

$$\begin{aligned} \{\phi_1, \phi_2\} &= -2\phi_1 + (P^M - A^M)\frac{\partial}{\partial X^M}(X \cdot A) - (P - A) \cdot A \\ &\quad - X^M \frac{\partial}{\partial X^M}[(P - A) \cdot A] - X^M \frac{\partial}{\partial X^M}(\frac{1}{2}A^2) \end{aligned} \quad (5.6b)$$

$$\{\phi_2, \phi_2\} = X^M F_{MN} X^N \quad (5.6c)$$

$$\{\phi_1, \phi_3\} = -\phi_2 \quad (5.6d)$$

$$\{\phi_2, \phi_3\} = -2\phi_3 \quad (5.6e)$$

$$\{\phi_3, \phi_3\} = 0 \quad (5.6f)$$

For the purposes of this paper, we see from the above equations that constraints (5.2)-(5.4) can be turned into first class constraints if the vector field  $A_M(X)$  satisfies the subsidiary conditions

$$F_{MN} = \frac{\partial A_N}{\partial X^M} - \frac{\partial A_M}{\partial X^N} = 0 \quad (5.7)$$

$$X.A = 0 \quad (5.8a)$$

$$(P - A).A = 0 \quad (5.8b)$$

$$\frac{1}{2}A^2 = 0 \quad (5.8c)$$

Condition (5.7) implies that the vector field  $A_M(X)$  defines a section of a flat  $U(1)$  bundle over the  $d + 2$  dimensional position space. In the case of vector fields for which  $F_{MN} \neq 0$ , condition (5.7) must be replaced by the subsidiary conditions

$$X^M F_{MN} = 0 \quad (5.9a)$$

$$(P^M - A^M)F_{MN} = 0 \quad (5.9b)$$

Condition (5.9a) appeared first in [5] but if we use it alone we are not taking into account the indistinguishability of  $X_M$  and  $P_M - A_M$  in the presence of the vector field  $A_M(X)$  for which  $F_{MN} \neq 0$ . As we see from brackets (5.6), both conditions (5.9) are necessary in this case.

Conditions (5.8a)-(5.8c) imply that constraints (5.2)-(5.4) do not form an irreducible [23] set of constraints for 2T physics with a vector field  $A_M(X)$ . Combining conditions (5.8a)-(5.8c) with constraints (5.2)-(5.4) we obtain the irreducible set of constraints [19]

$$\phi_1 = \frac{1}{2}P^2 \approx 0 \quad \phi_2 = X.P \approx 0 \quad \phi_3 = \frac{1}{2}X^2 \approx 0 \quad (5.10a)$$

$$\phi_4 = X.A \approx 0 \quad \phi_5 = P.A \approx 0 \quad \phi_6 = \frac{1}{2}A^2 \approx 0 \quad (5.10b)$$

The reappearance of constraints (5.10a) explains why the vector field  $A_M(X)$  of vanishing strength tensor must be present in 2T physics. As we saw in section three, in order to have a regular constraint surface associated to constraints (5.10a), the origin of position space (viewed as part of phase space) must be removed and this creates a non-trivial topology.

It can be verified that constraints (5.10) are all first class. The 2T action in the presence of the vector field  $A_M(X)$  can then be written as

$$\begin{aligned} S = \int d\tau [ & \dot{X}.P - (\frac{1}{2}\lambda_1 P^2 + \lambda_2 X.P + \frac{1}{2}\lambda_3 X^2 \\ & + \lambda_4 X.A + \lambda_5 P.A + \frac{1}{2}\lambda_6 A^2) ] \end{aligned} \quad (5.11)$$

where the 2T Hamiltonian is

$$H = \frac{1}{2}\lambda_1 P^2 + \lambda_2 X.P + \frac{1}{2}\lambda_3 X^2 + \lambda_4 X.A + \lambda_5 P.A + \frac{1}{2}\lambda_6 A^2 \quad (5.12)$$

It is important to mention here that, to arrive at action (5.11), no use was made of conditions (5.7) and (5.9). Only conditions (5.8) were used. Action (5.11) therefore gives a unified description of 2T physics with all kinds of position dependent vector fields. Those for which  $F_{MN} \neq 0$  or those for which  $F_{MN} = 0$ .

Action (5.11) is invariant under the Lorentz  $SO(d, 2)$  transformation with generator  $L_{MN} = X_M P_N - X_N P_M$

$$\delta X_M = \frac{1}{2}\omega_{RS}\{L_{RS}, X_M\} = \omega_{MR}X_R \quad (5.13a)$$

$$\delta P_M = \frac{1}{2}\omega_{RS}\{L_{RS}, P_M\} = \omega_{MR}P_R \quad (5.13b)$$

$$\delta A_M = \frac{\partial A_M}{\partial X_R}\delta X_R \quad (5.13c)$$

$$\delta\lambda_\varrho = 0, \quad \varrho = 1, 2, \dots, 6 \quad (5.13d)$$

under which  $\delta S = 0$ . It can be checked that  $L_{MN}$  has weakly vanishing Poisson brackets with the first class constraints (5.10), being therefore also gauge invariant in the presence of the vector field  $A_M(X)$ .

Action (5.11) also has the local infinitesimal invariance generated by the first class constraints (5.10)

$$\delta X_M = \epsilon_\varrho(\tau)\{X_M, \phi_\varrho\} = \epsilon_1 P_M + \epsilon_2 X_M + \epsilon_5 A_M \quad (5.14a)$$

$$\begin{aligned} \delta P_M &= \epsilon_\varrho(\tau)\{P_M, \phi_\varrho\} = -\epsilon_2 P_M - \epsilon_3 X_M - \epsilon_4 A_M \\ &\quad - \epsilon_4 X^N \frac{\partial A_N}{\partial X^M} - \epsilon_5 P^N \frac{\partial A_N}{\partial X^M} - \epsilon_6 A^N \frac{\partial A_N}{\partial X^M} \end{aligned} \quad (5.14b)$$

$$\delta A_M = \frac{\partial A_M}{\partial X^N}\delta X_N \quad (5.14c)$$

$$\delta\lambda_1 = \dot{\epsilon}_1 + 2\epsilon_2\lambda_1 - 2\epsilon_1\lambda_2 \quad (5.14d)$$

$$\delta\lambda_2 = \dot{\epsilon}_2 + \epsilon_3\lambda_1 - \epsilon_1\lambda_3 \quad (5.14e)$$

$$\delta\lambda_3 = \dot{\epsilon}_3 + 2\epsilon_3\lambda_2 - 2\epsilon_2\lambda_3 \quad (5.14f)$$

$$\delta\lambda_4 = \dot{\epsilon}_4 + \epsilon_3\lambda_5 - \epsilon_5\lambda_3 \quad (5.14g)$$

$$\delta\lambda_5 = \dot{\epsilon}_5 + \epsilon_2\lambda_5 - \epsilon_5\lambda_2 \quad (5.14h)$$

$$\delta\lambda_6 = \dot{\epsilon}_6 \quad (5.14i)$$

under which

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \frac{d}{d\tau}(\epsilon_\varrho \phi_\varrho) \quad (5.15)$$



Now the conserved charge, or the generator of the local transformations (5.14), depending on whether the equations of motion are satisfied or not, is the quantity  $Q = \epsilon_{\varrho} \phi_{\varrho}$  with  $\varrho = 1, 2, \dots, 6$ . This generalizes the local  $Sp(2, R)$  invariance (3.11) of 2T physics to the case when a vector field  $A_M(X)$  is present.

Hamiltonian (5.12) is invariant under the finite local scale transformations

$$X_M \rightarrow \exp\{\beta(\tau)\} X_M \quad (5.16a)$$

$$P_M \rightarrow \exp\{-\beta(\tau)\} P_M \quad (5.16b)$$

$$A_M \rightarrow \exp\{-\beta(\tau)\} A_M \quad (5.16c)$$

$$\lambda_1 \rightarrow \exp\{2\beta(\tau)\} \lambda_1 \quad (5.16d)$$

$$\lambda_2 \rightarrow \lambda_2 \quad (5.16e)$$

$$\lambda_3 \rightarrow \exp\{-2\beta(\tau)\} \lambda_3 \quad (5.16f)$$

$$\lambda_4 \rightarrow \lambda_4 \quad (5.16g)$$

$$\lambda_5 \rightarrow \exp\{2\beta(\tau)\} \lambda_5 \quad (5.16h)$$

$$\lambda_6 \rightarrow \exp\{2\beta(\tau)\} \lambda_6 \quad (5.16i)$$

We can use the local scale invariance (5.16) to again select the gauge where  $\beta = \frac{1}{2}P^2$ . In this gauge we have the bracket relations

$$\{P_M, P_N\} = 0 \quad (5.17a)$$

$$\{X_M, P_N\} = \bar{G}_{MN} \quad (5.17b)$$

$$\{X_M, X_N\} = -L_{MN} \quad (5.17c)$$

$$\{X_M, A_N\} = -P_M A_N - X_M P^S \frac{\partial A_N}{\partial X^S} \quad (5.17d)$$

$$\{P_M, A_N\} = -\frac{\partial A_N}{\partial X^M} + P_M P^S \frac{\partial A_N}{\partial X^S} \quad (5.17e)$$

$$\{A_M, A_N\} = A_M P^S \frac{\partial A_N}{\partial X^S} - A_N P^S \frac{\partial A_M}{\partial X^S} \quad (5.17f)$$

where  $\bar{G}_{MN}$  is given by (3.17). The bracket relations (5.17) must be used in the place of the Poisson brackets (3.9) and (5.5) when performing classical Hamiltonian dynamics in the momentum space background  $\bar{G}_{MN}$  with a vector field  $A_M(X)$ . In the transition to the quantized theory using the correspondence principle, brackets (5.17) will give the fundamental commutators for a formulation of quantum mechanics based on the scale invariant Hamiltonian (5.12) where the geometry depends on the momenta while the vector field depends on the positions. This new mixed formulation must be an equally valid one because, as we will see below, the brackets (5.17) preserve the form of the classical Hamiltonian equations of motion.

To reach a representation where both the geometry and the vector field are position dependent, we use the local scale invariance (5.16) to select a gauge where  $\beta = \frac{1}{2}X^2$ . In this gauge we have the bracket relations

$$\{P_M, P_N\} = L_{MN} \quad (5.18a)$$

$$\{X_M, P_N\} = G_{MN} \quad (5.18b)$$

$$\{X_M, X_N\} = 0 \quad (5.18c)$$

$$\{X_M, A_N\} = 0 \quad (5.18d)$$

$$\{P_M, A_N\} = -\frac{\partial A_N}{\partial X^M} + X_M A_N \quad (5.18e)$$

$$\{A_M, A_N\} = 0 \quad (5.18f)$$

where  $G_{MN}$  is given by (3.18). In the transition to the quantized theory brackets (5.18) are turned into the fundamental commutators for a position space formulation of quantum mechanics based on the same scale invariant Hamiltonian (5.12).

In terms of the Poisson brackets (3.9) and (5.5), the classical equations of motion in the presence of the vector field  $A_M(X)$  are

$$\dot{X}_M = \{X_M, H\} = \lambda_1 P_M + \lambda_2 X_M + \lambda_5 A_M \quad (5.19a)$$

$$\begin{aligned} \dot{P}_M &= \{P_M, H\} = -\lambda_2 P_M - \lambda_3 X_M - \lambda_4 A_M \\ &\quad - \lambda_4 X^N \frac{\partial A_N}{\partial X^M} - \lambda_5 P^N \frac{\partial A_N}{\partial X^M} - \lambda_6 A^N \frac{\partial A_N}{\partial X^M} \end{aligned} \quad (5.19b)$$

$$\dot{A}_M = \{A_M, H\} = \lambda_1 P^N \frac{\partial A_M}{\partial X^N} + \lambda_2 X^N \frac{\partial A_M}{\partial X^N} + \lambda_5 A^N \frac{\partial A_M}{\partial X^N} \quad (5.19c)$$

where  $H$  is given by (5.12). Since the Hamiltonian (5.12) is scale invariant, after dropping terms quadratic in the constraints (5.10), it has the same expression in the backgrounds  $\bar{G}_{MN}$  and  $G_{MN}$ . However, each background requires its own bracket structure. As can be verified, after dropping terms linear in the constraints (5.10), the equations of motion (5.19) remain invariant if we change to the momentum space background  $\bar{G}_{MN}$  and use the brackets (5.17). After dropping terms linear in the constraints (5.10), the equations of motion (5.19) also remain invariant if we change to the position space background  $G_{MN}$  while at the same time changing to the bracket relations (5.18). Up to now we may say that using the local scale invariance (5.16) we have uncovered three formulations of quantum mechanics in three different spaces and with  $A_M = A_M(X)$ . These three formulations have the same classical limit described by the 2T Hamiltonian (5.12).

Now we use the local indistinguishability of position and momentum in 2T physics and modify the 2T Hamiltonian (3.5) according to the new rule  $X_M \rightarrow X_M - A_M(P)$ . Action (3.4b) then becomes

$$S = \int d\tau [\dot{X} \cdot P - (\frac{1}{2}\lambda_1 P^2 + \lambda_2 (X - A) \cdot P + \frac{1}{2}\lambda_3 (X - A)^2)] \quad (5.20)$$

The equations of motion for the Lagrange multipliers now give the constraints

$$\phi_1 = \frac{1}{2}P^2 \approx 0 \quad (5.21)$$

$$\phi_2 = (X - A).P \approx 0 \quad (5.22)$$

$$\phi_3 = \frac{1}{2}(X - A)^2 \approx 0 \quad (5.23)$$

The Poisson brackets between the canonical variables and the vector field  $A_M(P)$  are

$$\{X_M, A_N\} = \frac{\partial A_N}{\partial P^M} \quad (5.24a)$$

$$\{P_M, A_N\} = 0 \quad (5.24b)$$

$$\{A_M, A_N\} = 0 \quad (5.24c)$$

Computing the algebra of constraints (5.21)-(5.23) using the Poisson brackets (3.9) and (5.24), we find the expressions

$$\{\phi_1, \phi_1\} = 0 \quad (5.25a)$$

$$\{\phi_1, \phi_2\} = -2\phi_1 \quad (5.25b)$$

$$\{\phi_1, \phi_3\} = -\phi_2 \quad (5.25c)$$

$$\{\phi_2, \phi_2\} = -P^M \bar{F}_{MN} P^N \quad (5.25d)$$

$$\begin{aligned} \{\phi_2, \phi_3\} = & -2\phi_3 - P_M \frac{\partial}{\partial P_M} [(X - A).A] - P_M \frac{\partial}{\partial P_M} \left(\frac{1}{2}A^2\right) \\ & + (X_M - A_M) \frac{\partial}{\partial P_M} (P.A) - (X - A).A \end{aligned} \quad (5.25e)$$

$$\{\phi_3, \phi_3\} = -(X^M - A^M) \bar{F}_{MN} (X^N - A^N) \quad (5.25f)$$

For the case in which we are interested in this paper, we see that constraints (5.21)-(5.23) can be turned into first class constraints if we impose the subsidiary conditions on the vector field  $A_M(P)$

$$\tilde{F}_{MN} = \frac{\partial A_N}{\partial P^M} - \frac{\partial A_M}{\partial P^N} = 0 \quad (5.26)$$

$$(X - A).A = 0 \quad (5.27a)$$

$$P.A = 0 \quad (5.27b)$$

$$\frac{1}{2}A^2 = 0 \quad (5.27c)$$

Condition (5.26) implies that the vector field  $A_M(P)$  defines a section of a flat U(1) bundle over the momentum space. For vector fields  $A_M(P)$  for which  $\bar{F}_{MN} \neq 0$  the subsidiary conditions can be obtained from brackets (5.25f) and (5.25d) and are

$$(X^M - A^M) \bar{F}_{MN} = 0 \quad (5.28a)$$

$$P^M \bar{F}_{MN} = 0 \quad (5.28b)$$

Compare conditions (5.28) with conditions (5.9) that were obtained in the case of a vector field  $A_M(X)$  for which  $F_{MN} \neq 0$ . There is a clear dual relation between (5.9) and (5.28).

Conditions (5.27a)-(5.27c) imply that constraints (5.21)-(5.23) do not form an irreducible set of constraints for 2T physics with a vector field  $A_M(P)$ . Combining conditions (5.27a)-(5.27c) with constraints (5.21)-(5.23) we arrive at the same set of irreducible first class constraints (5.10). Action (5.11) therefore gives a unified general description of 2T physics with position dependent or momentum dependent vector fields because again no use was made of conditions (5.26) and (5.28) to arrive at action (5.11).

To each of the symmetries (5.13), (5.14) and (5.16) of action (5.11) with  $A_M = A_M(X)$  there is a corresponding symmetry with  $A_M = A_M(P)$ . The global  $SO(d, 2)$  invariance with generator  $L_{MN} = X_M P_N - X_N P_M$  is given by the transformation equations

$$\delta X_M = \frac{1}{2} \omega_{RS} \{L_{RS}, X_M\} = \omega_{MR} X_R \quad (5.29a)$$

$$\delta P_M = \frac{1}{2} \omega_{RS} \{L_{RS}, P_M\} = \omega_{MR} P_R \quad (5.29b)$$

$$\delta A_M = \frac{\partial A_M}{\partial P_N} \delta P_N \quad (5.29c)$$

$$\delta \lambda_\varrho = 0 \quad \varrho = 1, 2, \dots, 6 \quad (5.29d)$$

under which  $\delta S = 0$ . It can be verified that  $L_{MN}$  has weakly vanishing Poisson brackets with constraints (5.10) when  $A_M = A_M(P)$ , being therefore also gauge invariant in this case.

The first class constraints (5.10) generate the local infinitesimal transformations

$$\begin{aligned} \delta X_M = \epsilon_\varrho(\tau) \{X_M, \phi_\varrho\} = \epsilon_1 P_M + \epsilon_2 X_M + \epsilon_4 X^S \frac{\partial A_S}{\partial P^M} \\ + \epsilon_5 A_M + \epsilon_5 P^S \frac{\partial A_S}{\partial P^M} + \epsilon_6 A^S \frac{\partial A_S}{\partial P^M} \end{aligned} \quad (5.30a)$$

$$\delta P_M = \epsilon_\varrho(\tau) \{P_M, \phi_\varrho\} = -\epsilon_2 P_M - \epsilon_3 X_M - \epsilon_4 A_M \quad (5.30b)$$

$$\delta A_M = \frac{\partial A_M}{\partial P_N} \delta P_N \quad (5.30c)$$

$$\delta \lambda_1 = \dot{\epsilon}_1 + 2\epsilon_2 \lambda_1 - 2\epsilon_1 \lambda_2 \quad (5.30d)$$

$$\delta \lambda_2 = \dot{\epsilon}_2 + \epsilon_3 \lambda_1 - \epsilon_1 \lambda_3 \quad (5.30e)$$

$$\delta \lambda_3 = \dot{\epsilon}_3 + 2\epsilon_3 \lambda_2 - 2\epsilon_2 \lambda_3 \quad (5.30f)$$

$$\delta \lambda_4 = \dot{\epsilon}_4 + \epsilon_4 \lambda_2 - \epsilon_2 \lambda_4 \quad (5.30g)$$

$$\delta \lambda_5 = \dot{\epsilon}_5 + \epsilon_4 \lambda_1 - \epsilon_1 \lambda_4 \quad (5.30h)$$

$$\delta\lambda_6 = \dot{\epsilon}_6 \quad (5.30i)$$

under which

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \frac{d}{d\tau} (\epsilon_\varrho \phi_\varrho) \quad (5.31)$$

The conserved Hamiltonian Noether charge  $Q$ , or the generator of the local transformations (5.30), depending on whether the equations of motion are satisfied or not, is again a combination of the first class constraints (5.10), with the exception that now  $A_M = A_M(P)$ .

In the case when  $A_M = A_M(P)$ , Hamiltonian (5.12) is invariant under the finite local scale transformations

$$X_M \rightarrow \exp\{\beta(\tau)\} X_M \quad (5.32a)$$

$$P_M \rightarrow \exp\{-\beta(\tau)\} P_M \quad (5.32b)$$

$$A_M \rightarrow \exp\{\beta(\tau)\} A_M \quad (5.32c)$$

$$\lambda_1 \rightarrow \exp\{2\beta(\tau)\} \lambda_1 \quad (5.32d)$$

$$\lambda_2 \rightarrow \lambda_2 \quad (5.32e)$$

$$\lambda_3 \rightarrow \exp\{-2\beta(\tau)\} \lambda_3 \quad (5.32f)$$

$$\lambda_4 \rightarrow \exp\{-2\beta(\tau)\} \lambda_4 \quad (5.32g)$$

$$\lambda_5 \rightarrow \lambda_5 \quad (5.32h)$$

$$\lambda_6 \rightarrow \exp\{-2\beta(\tau)\} \lambda_6 \quad (5.32i)$$

Again we can use the local scale invariance (5.32) to select a gauge where  $\beta = \frac{1}{2}P^2$ . After manipulations similar to the case when  $A_M = A_M(X)$ , we arrive at the bracket relations

$$\{P_M, P_N\} = 0 \quad (5.33a)$$

$$\{X_M, P_N\} = \bar{G}_{MN} \quad (5.33b)$$

$$\{X_M, X_N\} = -L_{MN} \quad (5.33c)$$

$$\{X_M, A_N\} = \frac{\partial A_N}{\partial P^M} + P_M A_N \quad (5.33d)$$

$$\{P_M, A_N\} = 0 \quad (5.33e)$$

$$\{A_M, A_N\} = 0 \quad (5.33f)$$

In the transition to the quantized theory using the correspondence principle, the brackets (5.33) will give the fundamental commutators for a formulation of quantum mechanics in the presence of gravity based on the scale invariant Hamiltonian (5.12) in a momentum space with a non-trivial topology. In this formulation the metric tensor and the vector field are both momentum dependent. If we assume that 1T physics is embedded [18] in 2T physics, the existence of the  $d + 2$  dimensional brackets (5.33) can be used to justify the complementation of the equations of quantum mechanics we proposed in section four. The

brackets (5.33) are the momentum space correspondents of the position space brackets (5.18) we obtained above in the case when  $A_M = A_M(X)$

We can also use the local scale invariance (5.32) to select a gauge where  $\beta = \frac{1}{2}X^2$ . In this case we arrive at the brackets

$$\{P_M, P_N\} = L_{MN} \quad (5.34a)$$

$$\{X_M, P_N\} = G_{MN} \quad (5.34b)$$

$$\{X_M, X_N\} = 0 \quad (5.34c)$$

$$\{X_M, A_N\} = \frac{\partial A_N}{\partial P^M} + X_M X^S \frac{\partial A_N}{\partial P^S} \quad (5.34d)$$

$$\{P_M, A_N\} = -X_M A_N - P_M X^S \frac{\partial A_N}{\partial P^S} \quad (5.34e)$$

$$\{A_M, A_N\} = A_M X^S \frac{\partial A_N}{\partial P^S} - A_N X^S \frac{\partial A_M}{\partial P^S} \quad (5.34f)$$

In the transition to the quantized theory, brackets (5.34) will give the fundamental commutators for another formulation of quantum mechanics based on the same scale invariant Hamiltonian (5.12). In this formulation the geometry depends on the position while the vector field is momentum dependent. This formulation is dual to the formulation that can be obtained from brackets (5.17).

In the case of a momentum dependent vector field  $A_M(P)$ , Hamiltonian (5.12) generates the equations of motion

$$\begin{aligned} \dot{X}_M &= \{X_M, H\} = \lambda_1 P_M + \lambda_2 X_M + \lambda_5 A_M \\ &+ \lambda_4 X^N \frac{\partial A_N}{\partial P^M} + \lambda_5 P^N \frac{\partial A_N}{\partial P^M} + \lambda_6 A^N \frac{\partial A_N}{\partial P^M} \end{aligned} \quad (5.35a)$$

$$\dot{P}_M = \{P_M, H\} = -\lambda_2 P_M - \lambda_3 X_M - \lambda_4 A_M \quad (5.35b)$$

$$\dot{A}_M = \{A_M, H\} = -\lambda_2 P^N \frac{\partial A_M}{\partial P^N} - \lambda_3 X^N \frac{\partial A_M}{\partial P^N} - \lambda_4 A^N \frac{\partial A_M}{\partial P^N} \quad (5.35c)$$

computed in terms of the Poisson brackets (3.9) and (5.24). The Hamiltonian and the equations of motion remain invariant if we change to the background  $\bar{G}_{MN}$  supplied with brackets (5.33). The Hamiltonian and the equations of motion also remain invariant if we change to the background  $G_{MN}$  supplied with brackets (5.34). Using the local scale invariance (5.32) we have now uncovered three other formulations of quantum mechanics in three different spaces and with  $A_M = A_M(P)$ . We must then conclude that there are six possible formulations of quantum mechanics. These six formulations have the same classical limit described by the 2T Hamiltonian (5.12).

## 6 Concluding remarks

Starting with the massless scalar relativistic particle, we provided evidence for the necessity for the concept of a momentum dependent metric tensor in quantum mechanics in the presence of gravity. Then we showed that position and momentum dependent metric tensors have a natural existence in 2T physics as a consequence of a local scale invariance of the Hamiltonian. Also as a consequence of this local scale invariance, we verified that the classical Hamiltonian equations of motion for 2T physics are identical in the backgrounds defined by these position dependent and momentum dependent metric tensors. This demonstrates their equivalence at the classical level in 2T physics.

Based on this equivalence of position and momentum dependent metric tensors at the classical level in 2T physics, we wrote down the equations that makes quantum mechanics in the presence of gravity in total agreement with the wave-particle duality. In constructing these equations for the case of momentum spaces with non-trivial topology, we had to introduce the concept of a momentum dependent vector field. As a basis for this concept, this paper verifies that the symmetries which are present in 2T physics when  $A_M = A_M(X)$  are also present when  $A_M = A_M(P)$ .

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